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ON THE STABILITY OF UNIFORM ROTATIONS OF A RIGID BODY AROUND THE PRINCIPAL AXIS*

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The stability of uniform rotations of a rigid body with a fixed point around the principal axis supporting the center of mass is investigated in the case when the moment of inertia relative to the rotation axis equals one of the two other principal moments of inertia. The motions studied correspond to the boundary of the domain wherein the necessary stability conditions are fulfilled and to the curve on which the Arnol'd—Moser determinant vanishes.

1. On the stability of the steady-state motions of Hamiltonian systems. We consider the steady-state motions of an autonomous Hamiltonian system with $m+2$ degrees of freedom and m ignorable coordinates. In recent years theorems have been proved /1,2/ extending to such motions a number of results obtained from a study of the equilibrium position of two-dimensional Hamiltonian systems /3,4/. The need for carrying out the corresponding proofs is due to the presence of additional difficulties caused by the dependence of the Hamiltonian on the cyclic constants. On this basis the Theorem 1 below is given with proof, although analogous statements (without proof) were made in certain papers, for example, in /5,6/.

Let the steady-state motion being studied correspond to a point P with coordinates

$$p_j = 0,\quad q_j = 0\quad (j = 1, 2),\quad p_{2+n} = c_n^\circ\quad (n = 1, \ldots, m) \tag{1.1}$$

and let the Hamiltonian H be an analytic function of its own variables at this point. If the quadratic part H_{c° of the Hamiltonian of the reduced system is a sign-definite function of its variables, then the Liapunov-stability of steady-state motion (1.1) follows from Routh's theorem with Liapunov's supplement. Suppose that H_{c° is not a sign-definite function of its variables. The following theorem on the equivalence of the stability of steady-state motion (1.1) and of the equilibrium position of the reduced system with Hamiltonian H_{c° holds.

Theorem 1. Let the eigenvalues of the linearized reduced system be pure imaginary at point P: $\pm i\alpha_1(c^\circ)$, $\pm i\alpha_2(c^\circ)$, and let the frequencies not be connected by a first-order resonance relation $k_1\alpha_1(c^\circ) + k_2\alpha_2(c^\circ) \neq 0$, $|k_1| + |k_2| = 1$ (k_1 and k_2 are integers). Then from the Liapunov-stability of the equilibrium position of the reduced system, proved by reduction with the application of Moser's theorem on mappings, follows the Liapunov-stability of steady-state motion (1.1).

Proof. Since the Hamiltonian H satisfies the conditions of Lemma 1.1 in /7/, a rest point of the reduced system corresponds to the steady-state motion for each c from some neighborhood of point c°, while the Hamiltonian H_c of the reduced system is an analytic function of the cyclic constants at point c° and can be presented as the series

$$H_c = H_2 + H_3 + \ldots + H_m + \ldots,\qquad H_m = \sum_{\nu=m} h_{\nu_1\nu_2\nu_3\nu_4} p_1^{\nu_1} p_2^{\nu_2} q_1^{\nu_3} q_2^{\nu_4}\quad (\nu = \nu_1 + \nu_2 + \nu_3 + \nu_4) \tag{1.2}$$

whose coefficients $h_{\nu_1\nu_2\nu_3\nu_4}(c)$ are analytic functions of c at point c°. To prove the theorem we need to investigate the perturbed motions

$$p_j = \varepsilon p_j',\quad q_j = \varepsilon q_j'\quad (j = 1, 2),\quad c_n = c_n^\circ + \delta c_n'\quad (n = 1, \ldots, m),\quad |c'| \leqslant 1 \tag{1.3}$$

where $|x|$ is the Euclidean norm of vector x, $\delta = \varepsilon^k$ (it is usual to assume $k = 1$). The choice of an appropriate value of k (a sufficiently small neighborhood of point c°) enables us to surmount the difficulties caused by the dependence of the coefficients of the reduced system's Hamiltonian on the cyclic constants. If the question on the stability of the equilibrium position of the reduced system with Hamiltonian H_{c° can be resolved by means of reduction (the investigation is led to a system with one degree of freedom, but nonautonomous; for example, see /3,8/) by forms up to order $2+\alpha$, inclusive, in expansion (1.2), then we choose $k = \alpha + 1$. Then, representing the functions $h_{\nu_1\nu_2\nu_3\nu_4}(c)$, analytic at point c°, by power series with due regard to the change of variables (1.3), we transform Hamiltonian (1.2) to

$$H_c = H_c^\circ(p_j', q_j', \varepsilon) + H_c^1(p_j', q_j', c_n', \varepsilon)\quad (j = 1, 2;\ n = 1, \ldots, m) \tag{1.4}$$

$$H_c^\circ = \sum_{i=2}^{2+\alpha} \sum_{\nu=i} \varepsilon^{i-2} h_{\nu_1\nu_2\nu_3\nu_4}(c^\circ)\, p_1'^{\nu_1} p_2'^{\nu_2} q_1'^{\nu_3} q_2'^{\nu_4}\ (\nu = \nu_1 + \nu_2 + \nu_3 + \nu_4)$$

Here H_c° is the unperturbed part of the Hamiltonian, while the perturbed part $H_c^1 = O(\varepsilon^{\alpha+1})$ is

*Prikl.Matem.Mekhan.,44,No.2,245-253,1980 uniformly bounded with respect to c' for all $|c'| \leqslant 1$. We note that the unperturbed part of Hamiltonian (1.4) is independent of c' and, consequently, $H_c^\circ = H_{c^\circ}^\circ$. Normalizing the unperturbed part of Hamiltonian (1.4) and using the integral $H_c = h$, we effect at the zero isoenergetic level a reduction to a one-dimensional system whose Hamiltonian is

$$K = (r\varepsilon)^\alpha \Phi(\varphi) + K^*(t, r, \varphi, c_n', \varepsilon) \quad (n = 1, \ldots, m)$$

where the functions Φ and $K^* = O(\varepsilon^{\alpha+1})$ are τ-periodic in φ and K^* is 2π-periodic in t and uniformly bounded in c' for all $|c'| \leqslant 1$. Here r and φ are the momentum and the coordinate of the one-dimensional system and t is a variable playing the role of time. The subsequent part of the proof coincides with the corresponding part of the proof of Theorem 2.1 in /8/. Applying Moser's mapping theorem /3/ to the reduced one-dimensional system, we get that the stability of steady-state motion (1.1) follows from the Liapunov-stability of the equilibrium position of the reduced system with Hamiltonian H_{c°.

Notes. 1°. Having proved in some fashion the instability of the rest point of a system with Hamiltonian H_{c°, by the same way we prove the instability of the steady-state motion (1.1).

2°. The requirement that Hamiltonian H be analytic at point P can be replaced by the condition that partial derivatives of order $\alpha + 8$ in all the arguments exist at this point. This follows from /9/ in which the requirement of analyticity of a mapping in Moser's theorem is replaced by the condition of existence of continuous fifth-order partial derivatives in all arguments, which is fulfilled if function H has continuous partial derivatives of order $\alpha + 8$ in all arguments at point P. The latter is sufficient also for obtaining the uniform upper estimates figuring in the proof of the theorem on remainder terms relative to c'.

The conditions for the stability of the equilibrium position of an autonomous Hamiltonian system with two degrees of freedom in the absence of resonances up to order $2n$, inclusive /10/, with resonances $\alpha_1 = 3\alpha_2, \alpha_1 = 2\alpha_2$ /4,11/, $\alpha_1 = \alpha_2$ and the elementary divisors are simple /12/, were obtained precisely by reduction with a subsequent application of Moser's mapping theorem to the reduced system. Therefore, according to Theorem 1 they remain valid for the steady-state motions of an autonomous Hamiltonian system whose reduced system is two-dimensional. In the absence of resonances up to fourth order, inclusive, and in the case of fourth-order resonance this result was obtained in /1,2/. In the case of equal frequencies $\alpha_1 = \alpha_2$ and nonsimple elementary divisors the stability conditions for the equilibrium position of a two-dimensional autonomous Hamiltonian system have been obtained in /13/ without passing to the reduced system. But in the given case the result can be obtained by means of reduction of the system to a one-dimensional one with a subsequent application of Moser's mapping theorem /14/ and, consequently, carries over to steady-state motions.

2. Stability of uniform rotations. Statement of the problem.

Let us describe the motion of a heavy rigid body whose center of mass lies on the principal axis by Hamiltonian equations. By directing the axes of the coordinates system connected with the body along the principal axes of the energy ellipsoid and introducing the Euler angles in the usual manner, we write an expression for the Hamiltonian in the assumption that the center of mass lies on the first principal axis

$$H = \frac{1}{2A_1A_2\sin^2\vartheta}\{A_1[p_\vartheta^2\sin^2\vartheta + (p_\psi - p_\varphi\cos\vartheta)^2] + (A_2 - A_1)\times$$

$$[p_\vartheta\sin\vartheta\cos\varphi + (p_\psi - p_\varphi\cos\vartheta)\sin\varphi]^2\} + \frac{p_\varphi^2}{2A_3} + \Gamma e\sin\vartheta\sin\varphi$$

Here A_1, A_2, A_3 are the body's principal moments of inertia for a fixed point; Γ is the product of the body's weight by the distance to the center of mass; $e = 1$ if the center of mass lies above the support point and $e = -1$ otherwise.

Uniform rotations around the first principal axis with angular velocity ω are determined by the following values of the variables:

$$p_\vartheta = 0,\ \ p_\varphi = 0,\ \ p_\psi = \omega A_1,\ \ \vartheta = \pi/2,\ \ \varphi = \pi/2,\ \ \varphi = \omega t + \psi_0 \tag{2.1}$$

The necessary stability conditions for uniform rotations (2.1) were obtained and analyzed in detail in /15/. The sufficient stability conditions were indicated in /16/. In the case of equality of the body's principal moments of inertia $A_1 = A_2$ with $e = -1$ the necessary stability conditions coincide with the sufficient. Let us consider the stability of the uniform rotations (2.1) in case $A_1 = A_2$ with $e = 1$. Using the integral $p_\psi = \mathrm{const}$, we pass to a reduced system with two degrees of freedom. Setting

$$p_\vartheta = x_1',\ \ p_\varphi = x_2',\ \ \vartheta = \pi/2 + y_1',\ \ \varphi = \pi/2 + y_2'$$

in the perturbed motion, we find the expansion of the reduced system's Hamiltonian in a neighborhood of its equilibrium position to within terms of sixth order relative to x_1', x_2', y_1', y_2'. After passing to a dimensionless time and dimensionless variables x_1, x_2, y_1, y_2 by the formulas

$$\tau = t\sqrt{\Gamma / A_1}, \quad x_j' = \sqrt{\Gamma A_1} x_j, \quad y_j' = y_j \quad (j = 1, 2)$$

the expansion of the reduced system's Hamiltonian is

$$H = H_2 + H_4 + H_6 + \ldots, \qquad H_m = \sum_{\nu=m} K_{\nu_1\nu_2\nu_3\nu_4} x_1^{\nu_1} x_2^{\nu_2} y_1^{\nu_3} y_2^{\nu_4} \quad (\nu = \nu_1 + \nu_2 + \nu_3 + \nu_4) \tag{2.2}$$

$$H_2 = \frac{1}{2} x_1^2 + \frac{b}{2} x_2^2 + \mu x_2 y_1 + \frac{1}{2}(\mu^2 - 1) y_1^2 - \frac{1}{2} y_2^2$$

$$H_4 = \frac{1}{2} x_2^2 y_1^2 + \frac{5}{6} \mu x_2 y_1^3 + \frac{8\mu^2 + 1}{24} y_1^4 + \frac{1}{4} y_1^2 y_2^2 + \frac{1}{24} y_2^4$$

$$H_6 = \frac{1}{3} x_2^2 y_1^4 + \frac{61}{120} \mu x_2 y_1^5 + \frac{136\mu^2 - 1}{720} y_1^6 - \frac{1}{48} y_1^2 y_2^2 (y_1^2 + y_2^2) - \frac{1}{720} y_2^6$$

$$b = A_1 / A_3, \quad \mu = \omega \sqrt{A_1 / \Gamma} = p_\psi / \sqrt{\Gamma A_1}$$

The triangle inequalities for the moments of inertia delineate in the plane $Ob\mu$ a domain $C\ (-\infty < \mu < +\infty;\ {}^1/_2 < b < +\infty)$ of admissible values of the dimensionless parameters.

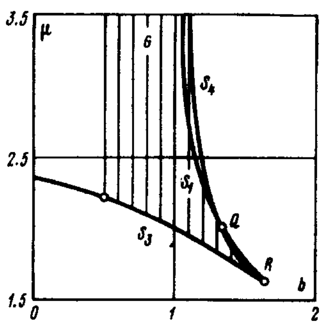

Fig.1

In the case being analyzed there is only the energy integral $H = \text{const}$ and the function H^2 is sign-constant; therefore, it is impossible to find sufficient stability conditions by constructing a Liapunov function from the integrals of the equations of perturbed motion /17/. In /1,2/ it was shown that in the subdomain $G\ (3 - b + 2\sqrt{2 - b} < \mu^2 < b / (b - 1);\ {}^1/_2 < b < (\sqrt{5} + 1) / 2)$ of domain C, in which only the necessary stability conditions are fulfilled, the uniform rotations (2.1) are stable everywhere except on a curve S_1 (Fig.1) at whose points the Arnol'd—Moser determinant /3/ vanishes. (All the constructions on the Fig .1 have been carried out for $\mu > 0$ since the graphs for $\mu < 0$ are symmetric with the ones indicated relative to axis Ob). Uniform rotations corresponding to points of the boundary curve $S_4\ (\mu = \sqrt{b / (b - 1)};\ 1 < b < (\sqrt{5} + 1) / 2)$ of domain G were studied in /18/. It was shown that the uniform rotations corresponding to the points of this curve are unstable for ${}^4/_3 < b < (\sqrt{5} + 1)/2$ (curve RQ) and are stable for fixed values of angular velocity for $1 < b < {}^4/_3$ Thus, besides the uniform rotations corresponding to points of curve S_1 only the uniform roations corresponding to point $Q\ (\mu = 2;\ b = {}^4/_3)$ of the boundary curve S_4 and to points of the boundary curve $S_3\ (\mu = [3 - b + 2(2 - b)^{1/2}]^{1/2};\ {}^1/_2 < b < (\sqrt{5} + 1) / 2)$ of domain G remained uninvestigated. The study of the stability of these uniform rotations completes the analysis of the stability problem for the uniform rotations (2.1) in the case of equality of the body's principal moments of inertia $A_1 = A_2$. To answer the question on stability on the determinant curve and at the point Q of boundary curve S_4 we need to normalize the Hamiltonian up to terms of order higher than fourth, since terms up to fourth order, inclusive, do not resolve the stability question /1,18/.

3. Investigation of stability of uniform rotations corresponding to points of determinant curve S_1.

Computing the frequencies of the linearized system (2.2)

$$\alpha_{1,2} = [{}^1/_2 (Q_1 \mp D^{1/2})]^{1/2}, \qquad (Q_1 = \mu^2 - b - 1, \quad D = \mu^4 + 2\mu^2 (b - 3) + (b - 1)^2)$$

we write the canonic transformation normalizing the quadratic part of the Hamiltonian /1/

$$x_1 = \alpha_1 s_1 q_1 - \alpha_2 s_2 q_2, \quad x_2 = \frac{1}{\sqrt{\alpha_1 \alpha_2}} (\alpha_2 s_2 p_1 + \alpha_1 s_1 p_2)$$

$$y_1 = -s_1 p_1 - s_2 p_2, \quad y_2 = \sqrt{\alpha_1 \alpha_2} (-s_2 q_1 + s_1 q_2)$$

$$s_1 = k\sqrt{\alpha_2 (\alpha_1^2 + b)}, \quad s_2 = k\sqrt{\alpha_1 (\alpha_2^2 + b)}$$

Here $p_j,\ q_j\ (j = 1, 2)$ are the new variables, k is an arbitrary constant. The transformation's valency $c = k^2 \alpha_1 \alpha_2 (\alpha_1^2 - \alpha_2^2)$. This transformation, nonsingular in domain G, takes Hamiltonian (2.2) into the form (see (1.2) for the representation for H_m

$$H = \frac{\alpha_1}{2} (p_1^2 + q_1^2) - \frac{\alpha_2}{2} (p_2^2 + q_2^2) + H_4 + H_6 + \ldots \tag{3.1}$$

Let us write out the nonzero coefficients of the fourth-order form and the nonzero coefficients, needed for the investigation, of the sixth-order form (a part of them are presented below; to obtain the coefficients $h_{\nu_2\nu_1\nu_4\nu_3}$ from the expressions for $h_{\nu_1\nu_2\nu_3\nu_4}$ we need to interchange the positions of α_1 and α_2)

$$24ch_{4000} = k^4 \alpha_2^2 [(8\mu^2 + 1)(\alpha_1^2 + b)^2 + 12\mu^2 - 20\mu^2 (\alpha_1^2 + b)]$$

$$6ch_{3100} = k^4 \mu \alpha_2 \sqrt{\alpha_1 \alpha_2} [(8\mu^2 + 1)(\alpha_1^2 + b) + 6(\alpha_1^2 + \alpha_2^2 + 2b) - 5(\alpha_1^2 + b)(\alpha_1^2 + 3\alpha_2^2 + 4b)]$$

$$4ch_{2200} = k^4\alpha_1\alpha_2\,\{\mu^2\,(8\mu^2+1) + 2\,[(\alpha_1^2+b)^2 + (\alpha_2^2+b)^2 + 4\mu^2] - 10\mu^2\,(\alpha_1^2+\alpha_2^2+2b)\}$$

$$24ch_{0040} = k^4\alpha_1^4\alpha_2^2\,(\alpha_2^2+b)^2,\quad 6ch_{0031} = -k^4\mu\sqrt{\alpha_1\alpha_2}\,\alpha_1^3\alpha_2^2(\alpha_2^2+b)$$

$$4ch_{0022} = k^4\mu^2\alpha_1^3\alpha_2^3,\quad 2ch_{2011} = -k^4\mu\sqrt{\alpha_1\alpha_2}\,\alpha_1\alpha_2^2\,(\alpha_1^2+b)$$

$$4ch_{2002} = k^4\alpha_1\alpha_2^3\,(\alpha_1^2+b)^2,\quad 2ch_{1120} = k^4\mu\sqrt{\alpha_1\alpha_2}\,\alpha_1^2\alpha_2\,(\alpha_2^2+b)$$

$$4ch_{2020} = -ch_{1111} = k^4\mu^2\alpha_1^2\alpha_2^2$$

$$720ch_{6000} = k^6\alpha_2^3\,(\alpha_1^2+b)[(136\mu^2-1)(\alpha_1^2+b)^2 + 240\mu^2 - 366\mu^2\,(\alpha_1^2+b)]$$

$$720ch_{0060} = -k^6\alpha_1^6\alpha_2^3\,(\alpha_2^2+b)^3,\quad 48ch_{4020} = -k^6\mu^2\alpha_1^2\alpha_2^3\,(\alpha_1^2+b)$$

$$48ch_{2040} = -k^6\mu^2\alpha_1^4\alpha_2^3\,(\alpha_2^2+b),\quad 8ch_{2220} = -k^6\mu^2\alpha_1^3\alpha_2^2\,(\alpha_2^2+b)$$

$$8ch_{2022} = -k^6\mu^2\alpha_1^3\alpha_2^4\,(\alpha_1^2+b),\quad 48ch_{4200} = k^6\alpha_1\alpha_2^2\,\{\mu^2\,(136\mu^2-1)(\alpha_1^2+b) + 16\,[6\mu^2\,(\alpha_2^2+b) + 8\mu^2\,(\alpha_1^2+b) + (\alpha_1^2+b)^3] - 122\mu^2\,[2\mu^2 + (\alpha_1^2+b)^2]\}$$

$$48ch_{4002} = -k^6\alpha_1\alpha_2^4\,(\alpha_1^2+b)^3,\quad 48ch_{0240} = -k^6\alpha_1^5\alpha_2^2\,(\alpha_2^2+b)^3,\qquad 48ch_{0042} = -k^6\mu^2\alpha_1^5\alpha_2^4\,(\alpha_2^2+b).$$

At points of curve S_1 the frequencies α_1 and α_2 of the linearized system are not connected by resonance relations up to sixth order, inclusive. At the points of this curve we reduce Hamiltonian (3.1), using the Birkhoff transformation, to normal form, restricting ourselves to sixth-order terms. Since in the original problem odd-order forms are not present in the expansion of the Hamiltonian, fifth-order terms do not appear in the normalization of H_4, but the coefficients of form H_6 are changed. Therefore, the normalization is carried out in two stages: at first we normalize H_4 and we compute the changed coefficients of form H_6, and next we normalize H_6. We obtain

$$H = \frac{\alpha_1}{2}(p_1^2+q_1^2) - \frac{\alpha_2}{2}(p_2^2+q_2^2) + \sum_{i+j=2}^{3} c_{ij}\,(p_1^2+q_1^2)^i\,(p_2^2+q_2^2)^j + \ldots$$

The coefficients c_{20}, c_{11}, c_{02} of the fourth-order form were found in /1/. We write them as follows:

$$c_{20} = \frac{1}{8}(3h_{4000} + 3h_{0040} + h_{2020}),\qquad c_{11} = \frac{1}{4}(h_{2200} + h_{2002} + h_{0220} + h_{0022})$$

Introducing the notation $f_{\nu_1\nu_2\nu_3\nu_4} = h_{\nu_1\nu_2\nu_3\nu_4} + h_{\nu_2\nu_1\nu_4\nu_3}$ and $g_{\nu_1\nu_2\nu_3\nu_4} = h_{\nu_1\nu_2\nu_3\nu_4} - h_{\nu_2\nu_1\nu_4\nu_3}$ for the coefficients of the normal sixth-order form, we obtain

$$c_{30} = \frac{1}{16}(5f_{6000} + f_{4020}) - \frac{1}{32\alpha_1}[g_{3010}^2 + 4f_{3010}^2 + 16g_{4000}^2 + (f_{4000} - h_{2020})^2] + \frac{1}{128}\left(\frac{u_1}{3\alpha_1+\alpha_2} - \frac{v_1}{3\alpha_1-\alpha_2} + \frac{u_2}{\alpha_1+\alpha_2} - \frac{v_2}{\alpha_1-\alpha_2}\right)$$

$$c_{21} = \frac{1}{16}(3f_{4200} + 3f_{4002} + f_{2220}) - \frac{3}{16\alpha_1}[f_{3010}f_{1210} + 2g_{4000}(g_{2200} + g_{2002})] + \frac{1}{16\alpha_2}(f_{2101}^2 + v_4^2) - \frac{9}{128}\left(\frac{u_1}{3\alpha_1+\alpha_2} + \frac{v_1}{3\alpha_1-\alpha_2}\right) + \frac{1}{128(\alpha_1+\alpha_2)}\{-3u_2 + 4u_3 + 2[(g_{1210}+g_{0121})^2 + (u_4+h_{1111})^2]\} - \frac{1}{128(\alpha_1-\alpha_2)}\{3v_2 + 4v_3 + 2[(g_{1210}-g_{0121})^2 + (u_4-h_{1111})^2]\}$$

$$u_1 = (f_{3001} - f_{1021})^2 + (g_{3100} - g_{1120})^2,\qquad u_2 = (3f_{3100} + f_{1120})^2 + (3g_{3001} + g_{1021})^2$$

$$u_3 = (3f_{3100} + f_{1120})(3f_{1300} + f_{1102}) + (3g_{3001} + g_{1021})(3g_{1003} + g_{1201})$$

$$u_4 = f_{2200} - f_{2002}$$

The formulas for v_1, v_2, v_3, v_4 are found from the expressions for u_1, u_2, u_3, u_4 by replacing $f_{\nu_1\nu_2\nu_3\nu_4}$ by $g_{\nu_1\nu_2\nu_3\nu_4}$ and *vice versa*. To obtain the coefficients c_{02}, c_{03}, c_{12} from the expressions for c_{20}, c_{30}, c_{21} we need to interchange the positions of α_1 and $-\alpha_2$ and the indices in coefficients $h_{\nu_1\nu_2\nu_3\nu_4}$: the first and second, the third and fourth.

To study the stability of the uniform rotations corresponding to points of curve S_1 we apply Theorem 2.1 proved in /10/ and extended to steady-state motions by using Theorem 1. The determinant curve S_1 is defined by the equation

$$D_2 = c_{20}\alpha_2^2 + c_{11}\alpha_2\alpha_1 + c_{02}\alpha_1^2 = 0$$

The uniform rotations corresponding to points of curve S_1, for which

$$D_3 = c_{30}\alpha_2^3 + c_{21}\alpha_2^2\alpha_1 + c_{12}\alpha_2\alpha_1^2 + c_{03}\alpha_1^3 \neq 0$$

are stable. We do not write out the expression for $D_3(\mu, b)$ because it is cumbersome. The equation $D_3 = 0$ was analyzed on a computer and this showed that the curves defined by this equation do not intersect curve S_1 in domain G (it has been established analytically that intersection obtains only at points R and Q of the boundary of domain G). Consequently, the uniform rotations corresponding to points of curve S_1 from domain G are stable.

4. Stability of uniform rotations corresponding to second-order resonance. On the boundary curve S_3 the frequencies α_1 and α_2 of the system linearized in a neighborhood of the motions being studied are connected by a second-order resonance relation: $\alpha_1 = \alpha_2 = \alpha \neq 0$. The linear canonic transformation

$$x_1 = \frac{\mu}{2}(K_1p_1 - q_2),\quad x_2 = \frac{\mu}{2\alpha}(K_1p_2 - q_1),\quad y_1 = \frac{\mu}{2\alpha}(q_1 - K_2p_2),\quad y_2 = \frac{\mu}{2}(q_2 - K_2p_1)$$

$$K_1 = \frac{1}{2\alpha} + \frac{\alpha}{\mu},\quad K_2 = \frac{1}{2\alpha} - \frac{\alpha}{\mu},\quad \alpha = \sqrt{\mu^2 - \mu - 1}$$

of valency $c = 2/\mu$, with a subsequent normalization of the fourth-order form , leads Hamiltonian (2.2) to the form

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \alpha(q_1p_2 - q_2p_1) + (q_1^2 + q_2^2)[A(q_1^2 + q_2^2) + B(q_1p_2 - q_2p_1) + C(p_1^2 + p_2^2)] + \ldots$$

We take advantage of the results in /13,14/, extended by use of Theorem 1 to steady-state motions. Since

$$A = \frac{\mu^4 - 2\mu^3 + 9\mu^2 - 20\mu + 12}{64\alpha^4c^3} > 0$$

at each point of curve S_3 , we conclude that the uniform rotations corresponding to the points of curve S_3 are stable.

5. Stability of uniform rotations corresponding to point Q. First-order resonance (one frequency is zero) holds on the boundary curve S_4. In this case, the linear canonic transformation found in /18/

$$x_1 = \frac{1}{\alpha^2}[-\mu p_1 + \sqrt{\alpha}(\mu^2 - 1)p_2],\quad x_2 = \frac{1}{\alpha^2}\left[(\mu^2 - 1)q_1 + \frac{\mu}{\sqrt{\alpha}}q_2\right]$$
$$y_1 = -\frac{1}{\alpha^2}\left[\mu q_1 + \frac{\mu^3 - 1}{\sqrt{\alpha}}q_2\right],\quad y_2 = \frac{1}{\alpha^2}[(\mu^2 - 1)p_1 - \mu\sqrt{\alpha}p_2]$$
$$\alpha = \{[(\mu^2 - 1)^2 - \mu^2]/(\mu^2 - 1)\}^{1/2}$$

of valency $c = -(\mu^2 - 1)/\alpha^2$ takes Hamiltonian (2.2) to the form (see (1.2) for the representation for H_m)

$$H = \frac{1}{2}p_1^2 - \frac{\alpha}{2}(p_2^2 + q_2^2) + H_4 + H_6 + \ldots \tag{5.1}$$

We write out the coefficients, needed for the stability investigation, of the fourth- and sixth-order forms

$$h_{0040} = \frac{\mu^2(\mu^2 - 4)}{8\alpha^6(\mu^2 - 1)},\quad h_{1030} = 0,\quad h_{0130} = 0,\quad h_{0031} = \frac{\mu(\mu^6 - 6\mu^4 + 4\mu^2 + 6)}{6\alpha^6\sqrt{\alpha}(\mu^2 - 1)},\quad h_{0060} = -\frac{\mu^4(2\mu^4 - 23\mu^2 + 48)}{144\alpha^{10}(\mu^2 - 1)}$$

Normalizing by Birknoff's transformation the fourth- and sixth-order forms in expansion (5.1) we find

$$H = \frac{1}{2}p_1^2 - \frac{\alpha}{2}(p_2^2 + q_2^2) + l_{40}q_1^4 + l_{22}q_1^2(p_2^2 + q_2^2) +$$
$$l_{04}(p_2^2 + q_2^2)^2 + l_{60}q_1^6 + l_{42}q_1^4(p_2^2 + q_2^2) + l_{24}q_1^2(p_2^2 + q_2^2)^2 + l_{06}(p_2^2 + q_2^2)^3 + \ldots$$

At point Q $(\mu = 2,\ b = 4/3)$

$$l_{40} = h_{0040} = 0,\qquad l_{60} = h_{0060} + \frac{4}{3}h_{0040}h_{2020} - \frac{1}{2}h_{1030}^2 + \frac{1}{2\alpha}(h_{0130}^2 + h_{0031}^2) = 0.05184 > 0$$

Making use of Theorem 4.1 of /8/, we conclude that the uniform rotations corresponding to point Q of boundary curve S_4 of domain G are stable for fixed values of the angular velocity (of the cyclic constant p_ψ).

Summing up, we state

Theorem 2. Suppose that a rigid body having equal moments of inertia relative to the first two axes rotates uniformly around the first axis supporting the center of mass, and that the center of mass is located above the support point. Such uniform rotations are stable everywhere in domain G and on the boundary curve S_3, are stable for fixed values of the angular velocity on a part $(1 < b \leqslant 4/3)$ of the boundary curve S_4, and are unstable on the remaining part $(4/3 < b < (\sqrt{5} + 1)/2)$ of curve S_4.

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